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Local polynomial convexity of tangential unions of totally real graphs in $\mathbf{C}^2$

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ABSTRACT

We give sufficient conditions so that the union of a totally real graph M in $\mathbf{C}^2$ and its tangent plane at 0 is locally polynomially convex at 0. Some examples of the non locally polynomially convex situation are also established.

1. INTRODUCTION

Let Y be a compact subset of $\mathbf{C}^n$ and let $\hat{Y}$ denote the polynomial hull of Y, i.e.

$$\hat{Y} = \{z \in \mathbf{C}^n : |Q(z)| \leq \max_Y |Q|, \text{ for every polynomial } Q \text{ on } \mathbf{C}^n\}.$$

We say that Y is polynomially convex if $\hat{Y} = Y$. A closed subset F of $\mathbf{C}^n$ is called locally polynomially convex (LPC) at $a \in F$ if there exists $r > 0$ such that $\overline{\mathbf{B}}(a, r) \cap F$ is polynomially convex. In the case F is a totally real, two dimensional, C^1 submanifold of $\mathbf{C}^2$, F is LPC everywhere [We, Chapter 17]. When F is the union of two real planes, a complete answer has been given by Weinstock [W].

Here we shall study the local polynomial convexity at 0 of a special case of the union of two totally real, 2-dimensional submanifolds of $\mathbf{C}^2$, M_1 and M_2, with the same tangent space at the origin. Several instances of such a situation, motivated by questions of local approximation, were studied by O'Farrell and De Paepe [D1], [D2], [D3], [D4], [DO].

Note that any totally real, real-analytic, two dimensional submanifold of $\mathbf{C}^2$

can be transformed into $\{w = \bar{z}\}$ by a local biholomorphic change of variables (see e.g. [MW]), thus we choose the following special setting. Let

$$(*) \qquad M_1 = \{(z, \bar{z}) : z \in D\}; \quad M_2 = \{(z, \bar{z} + \varphi(z)) : z \in D\},$$

where φ is a C^1 function in a neigbourhood of 0, denoted as $\varphi \in C^1\{0\}$, verifying $\varphi(0) = \partial\varphi/\partial z(0) = \partial\varphi/\partial\bar{z}(0) = 0$. Our results will be in terms of φ.

If we express in this setting the real-analytic case of the results of [DO] and [D4], we find that $M_1 \cup M_2$ is LPC at 0 for fome functions φ verifying (a) $\varphi(z) = a_1|z|^2 + a_2\bar{z}^2 + O(|z|^3)$, with $|a_1| > |a_2|$, or (b) $\varphi(z) = -2iz^{n+m}\bar{z}^{m+1} + O(|z|^{2m+n+2})$, where m is an integer and n is a nonnegative odd integer satisfying $2m + n \geq 1$.

The outline of the paper is as follows. In Proposition 2.1, we consider the case where φ can be represented by a Taylor series and give a sufficient condition on the lowest order coefficients of the series so that the union of $M_1 \cup M_2$ is LPC at 0. This in particular generalizes the results mentioned above (in the real-analytic case).

Proposition 2.3 is stated for the slightly more general class of closed sets having only a finite number of points lying 'above' any point of $\mathbf{C} \times \{0\}$; if we further assume that these sets are 'degenerate' i.e., that there exists a holomorphic polynomial mapping this set onto a 'thin' enough set in $\mathbf{C}$, a result of Stolzenberg gives sufficient conditions for polynomial convexity.

We show that some conditions of Proposition 2.1 cannot be dropped in Proposition 2.2. The study of this family of simple examples is completed using Proposition 2.3.

2 RESULTS

We always take the manifolds M_1 and M_2 of the form $(*)$. We denote, for each $r > 0$, $M_j^r := M_j \cap \{(z, w) : |z| \leq r\}, j = 1, 2$.

Proposition 2.1. *Suppose that φ is a function of the form*

$$\varphi(z) = \sum_{k=0}^{m} a_k \bar{z}^k z^{m-k} + \psi(z) =: \varphi_1(z) + \psi(z),$$

where $\psi \in C^1$ near the origin, $m \geq 2$, $\psi(z) = O(|z|^{m+1})$. If there exists $0 \leq 1 \leq [m/2]$ such that

$$(1) \qquad \sum_{j \neq l} |a_j| < |a_l|,$$

then $M_1 \cup M_2$ is LPC at 0.

Proof. The following result was given in [K] in a slighly more restrictive form, see [St, p. 386] (also [W]) for a proof of the present statement.

Kallin's Lemma. *If K and L are two polynomially convex compact sets in $\mathbf{C}^n$ and* *if we can find a polynomial p which sends K to the real line and L to a compact set meeting the real line only at the origin; if furthermore $p^{-1}\{0\} \cap (K \cup L)$ is a polynomially convex, then $K \cup L$ is polynomially convex.*

It follows from (1) that we can find $\lambda \in \mathbf{C}$ such that

$$(2) \qquad |\mathrm{Im}\,(\lambda a_l)| > |\lambda| \sum_{j \neq l} |a_j|.$$

We put $p(x,y) = \bar{\lambda} x^{m-2l+1} + \lambda y^{m-2l+1}$. It is clear that p maps M_1 to $\mathbf{R}$. Now we claim that for r small enough $p(M_2^r) \cap \mathbf{R} = \{0\}$. Indeed,

$$\begin{aligned}\mathrm{Im}\, p(z, \bar{z} + \varphi(z)) &= \mathrm{Im}\,(\bar{\lambda} z^{m-2l+1} + \lambda(\bar{z} + \varphi_1(z))^{m-2l+1}) + O(|z|^{2m-2l+1}) \\ &= \mathrm{Im}\,(\lambda(m-2l+1)\bar{z}^{m-2l}\varphi_1(z)) + O(|z|^{2m-2l+1}),\end{aligned}$$

since $\bar{\lambda} z^{m-2l+1} + \lambda \bar{z}^{m-2l+1} \in \mathbf{R}$. Therefore,

$$\begin{aligned}|\mathrm{Im}\, p(z, \bar{z} + \varphi(z))| &\geq (m-2l+1)|z|^{2m-2l}\left(|\mathrm{Im}\,(\lambda a_l)| - |\lambda| \sum_{j \neq l} |a_j|\right) \\ &\quad + O(|z|^{2m-2l+1}) > 0\end{aligned}$$

for any $z \neq 0$ in a small enough neighborhood of 0, by (2).

By Kallin's Lemma, it suffices to check that the set $p^{-1}(0) \cap (M_1^r \cup M_2^r)$ is polynomially convex for r small enough. Obviously $p^{-1}(0) \cap M_1^r$ is polynomially convex for r small enough. On the other hand, from the preceding estimate we see that $p^{-1}(0) \cap M_2^r = \{0\}$ where r is small enough. Thus $p^{-1}(0) \cap (M_1^r \cup M_2^r)$ is polynomially convex for r small enough. □

Remark. The referee kindly observed to me that Proposition 2.1 is closely related to a result of de Paepe [D4, p. 89] where a similar use of Kallin's Lemma was made.

Our next proposition shows that if we replace the strict inequality in (1) by equality or if $l > m/2$ we may get nontrivial hull. As is frequently the case, this hull is foliated by one-dimensional analytic varieties.

Notation. From now on we denote by π the projection on the first coordinate axis ($\pi(z,w) = z$).

Proposition 2.2.

(a) *Let $\varphi(z) = z^p \bar{z}^q + z^q \bar{z}^p$, where (p,q) are natural numbers. Then $M_1 \cup M_2$ is LPC at 0 if and only if $|p-q| \leq 1$.*

(b) *Let $\varphi(z) = z^p \bar{z}^{p+1}$ where $p \geq 1$. Then $M_1 \cup M_2$ is not LPC at 0.*

Proof (a). For each $t > 0$, let $V_t := \{(z,w) \in \mathbf{C}^2 : z + w = t\}$. Put

$$\begin{aligned}K_t &:= V_t \cap M_1 = \{(z,\bar{z}) : 2\mathrm{Re}\, z = t\}, \text{ and} \\ L_t &:= V_t \cap M_2 = \{z, \bar{z} + \varphi(z)) : 2\mathrm{Re}\, z + z^p \bar{z}^q + \bar{z}^p z^q = t\}.\end{aligned}$$

Since V_t is a graph over $\mathbf{C} \times \{0\}$, consider $\pi(K_t) \cap \pi(L_t) = \pi(K_t \cap L_t)$. In polar coordinates $z = \rho e^{i\theta}$, the equation $z^p \bar{z}^q + \bar{z}^p z^q = 0$ reduces to $\cos((p-q)\theta) = 0$.

When $|p-q| \geq 2$ we see that $\pi(K_t) \cap \pi(L_t)$ has at least 2 points. By the maximum modulus principle, $M_1^r \cup M_2^r$ is never polynomially convex for any r small enough, because its hull contains an open subset of V_t bounded by a closed curve included in $K_t \cup L_t$ when t is small enough.

When $p = q$, we conclude from Proposition 2.1 that $M_1 \cup M_2$ is LPC at 0. The case $|p-q| = 1$ will be settled after the proof of Proposition 2.3.

(b). For each $t > 0$, let $W_t = \{(z, w) : zw = t\}$. Consider the sets

$$P_t := W_t \cap M_1 = \{(z, \bar{z}) : |z| = t^{1/2}\} \text{ and}$$
$$Q_t := W_t \cap M_2 = \{(z, \bar{z} + \varphi(z)) : |z| = t'\},$$

where t' is the unique positive solution of the equation $t'^2 + t'^{2p+2} = t$. Once again from the maximum modulus principle we see that the polynomial convex hull of $M_1^r \cup M_2^r$ will contain an open subset of W_t bounded by two closed curves P_t and Q_t for any $t > 0$ small enough and hence $M_1 \cup M_2$ is not LPC at 0. □

Remarks. (1) We should observe that in case (a) the intersection of M_1 and M_2 is rather big (contains real lines). However, in case (b) the intersection reduces to the origin.

(2) One could add to Proposition 2.2(b) the case where $\varphi = az^p \bar{z}^{p+1}$ where $p \geq 1$ and a is a complex, non-real, number. Then, using the polynomial $p(x, y) = xy$ and Kallin's Lemma we may conclude that $M_1 \cup M_2$ is LPC at 0. Thus this example provides a class of functions φ to which Proposition 2.1 does not apply but the union $M_1 \cup M_2$ is still LPC at 0. The author is grateful to the referee for this observation.

To formulate the next result, we need to define the following property, which is clearly satisfied by $M_1^r \cup M_2^r$.

Definition. A compact set $K \subset \mathbf{C}^2$ is light if for every $z \in \mathbf{C}$ the set $\pi^{-1}(z) \cap K$ is finite.

Proposition 2.3. *A light set K in $\mathbf{C}^2$ is polynomially convex if there exists a polynomial p on $\mathbf{C}^2$ satisfying:*

(a) *p maps K to a simply connected set γ in $\mathbf{C}$ with empty interior.*

(b) *The set $\pi(p^{-1}(t) \cap K)$ is simply connected with empty interior for every $t \in \gamma$.*

If $H^1(K, \mathbf{Z}) = 0$ then (a) *may be weakened to* (a′)*: 'γ is the boundary of the unbounded component of $\mathbf{C} \backslash \gamma$'.*

Here π is the projection as above, $H^1(K, \mathbf{Z})$ denotes the first Cech cohomology group with integer coefficients of K.

Proof. First we prove K is polynomially convex under the hypotheses (a), (b). For each point $t \in \gamma$, we let $S_t := p^{-1}(t) \cap K$.

We claim that S_t is polynomially convex for every $t \in \gamma$. Indeed, from (b) we deduce that the set $\pi(S_t)$ is a simply connected compact set with empty interior. Hence, by Mergelyan's theorem (cf. [St], [G]), $\mathcal{P}(\pi(S_t)) = \mathcal{C}(\pi(S_t))$.

Thus, if $S \subset S_t$ is a set of antisymmetry of $\mathcal{P}(S_t)$ i.e., if $f \in \mathcal{P}(S_t)$ and f real on S imply that f is constant on S, then $\pi(S)$ consists of at most one point. Since K is light, S is finite. Thus Bishop's generalized Stone-Weierstrass theorem [St, p. 115], also called Bishop antisymmetry decomposition theorem [G, p. 60] gives $\mathcal{P}(S_t) = \mathcal{C}(S_t)$. In particular $\hat{S}_t = S_t$, q.e.d.

Now we are able to prove that K is polynomially convex. Since γ is simply-connected we have $\hat{\gamma} = \gamma$ and hence

$$\gamma \subset p(\hat{K}) \subset p(K)\hat{} = \hat{\gamma} = \gamma.$$

Lemma 2.4. ([St, p. 410], [Sg2]) *Let X be a compact subset of $\mathbf{C}^n$, p a polynomial, Ω the unbounded component of $\mathbf{C}\setminus p(X)$. If $\xi \in \partial\Omega$, then $p^{-1}(\xi) \cap \hat{X} = (p^{-1}(\xi) \cap X)\hat{}$.*

Applying Lemma 2.4 to $X = K$ and $t \in \gamma$, we get

$$p^{-1}(t) \cap \hat{K} = \hat{S}_t = S_t = p^{-1}(t) \cap K.$$

Thus K is polynomially convex.

It remains to establish the polynomial convexity of K under the hypotheses (a') and (b) when $H^1(K, \mathbf{Z}) = 0$. As before, S_t is polynomially convex for each $t \in \gamma$.

Lemma 2.5. ([Sg1, p. 279] *or* [St, p. 401]). *If K is a compact set in $\mathbf{C}^n$ with $H^1(K, \mathbf{Z}) = 0$, and if there is a polynomial p such that $p(K) \cap (p(\hat{K}\setminus K)) = \emptyset$ then K is polynomially convex.*

Suppose $\gamma \cap (p(\hat{K}\setminus K)) \neq \emptyset$. Let $t \in \gamma \cap p(\hat{K}\setminus K)$. Lemma 2.4 then gives

$$p^{-1}(t) \cap \hat{K} = \hat{S}_t = S_t = p^{-1}(t) \cap K, \text{ a contradiction. } \square$$

End of Proof of Proposition 2.2(a). If $|p - q| = 1$, say $q - p = 1$, consider the polynomial $p(x, y) = x + y$. By using the Proposition 2.3 it is enough to check that for r small enough $\pi(p^{-1}(t) \cap (M_1^r \cup M_2^r))$ is simply-connected with empty interior. We have

$$\pi(p^{-1}(t) \cap M_1^r) = \{z : (z, \bar{z}) \in M_1^r, \mathrm{Re}\, z = t/2\}$$
$$\pi(p^{-1}(t) \cap M_2^r) = \{z : (z, \bar{z} + \varphi(z)) \in M_2^r, 2\mathrm{Re}\, z(|z|^{2p} + 1) = t\}.$$

Clearly these two sets are disjoint for $t \neq 0$ and the first set is simply connected with empty interior. It remains to check that the other is so. Write $z = x + iy$, then

$$2\mathrm{Re}\, z(|z|^{2p} + 1) = t \iff 2x((x^2 + y^2)^p + 1) = t.$$

Thus x can be written as a function of y, hence $\pi(p^{-1}(t) \cap M_2^r)$ is simply-connected with empty interior. $\square$

Finally, we give another application of Proposition 2.3. To simplify notations, we define

$$\psi_1(t) = \begin{cases} t^3 \sin(1/t), & t \in \mathbf{R}\backslash\{0\} \\ 0, & t = 0, \end{cases}$$

$$\psi_2(z) = z + \bar{z} + (z + \bar{z})^2, \quad p_n(z, w) = z^n + w^n.$$

It is easy to see that $\psi_1 \in C^1(\mathbf{R})$ and $\psi_1'(0) = 0$.

Corollary 2.6. *The union $M_1 \cup M_2$ is LPC at 0 if φ is either*

(a) $$\varphi(z) = (z + \bar{z})^2 + i(\psi_1 \circ \psi_2)(z),$$

or

(b) $$\varphi(z) = \begin{cases} \bar{z}[(1 + \frac{g(z^n + \bar{z}^n)}{\bar{z}^n})^{1/n} - 1], & z \neq 0 \\ 0, & z = 0, \end{cases}$$

where g is a function of class C^1 in a neigbourhood of $0 \in \mathbf{R}$ and satisfies $g(0) = \frac{\partial g}{\partial x}(0) = 0$, such that either $g(x) \in \mathbf{R}$ for $|x| < r$ or $\operatorname{Im} g(x) \neq 0$ for $0 < |x| < r$, for some $r > 0$.

Proof (a). Here $p_1(M_1) = \mathbf{R}$ and, by the definition of φ, $p_1(M_2)$ is a smooth graph that intersects the real line at a sequence of points tending to the origin. So each point of the set $p_1(M_1 \cup M_2)$ lies in the boundary of the unbounded component of the complement of this set.

Next for each $t \in p_1(M_1) \cup p_1(M_2)$ we set $S_t := p_1^{-1}(t) \cap (M_1 \cup M_2)$. It is not hard to see that $\pi(S_t)$ is the union of at most three lines parallel to the imaginary axis. Hence $\pi(S_t)$ is simply connected with empty interior.

For r small enough, π restricted to each M_j^r is a homeomorphism onto a disk, and $\pi(M_1^r \cap M_2^r) = \{iy : -r \leq y \leq r\}$, hence $M_1^r \cup M_2^r$ is contractible, and its cohomology vanishes. So by Proposition 2.3 $M_1 \cup M_2$ is LPC at 0.

(b). From the definition of φ,

(3) $$p_n(z, \bar{z} + \varphi(z)) = z^n + \bar{z}^n + g(z^n + \bar{z}^n).$$

Then p_n sends M_1^r onto a compact interval γ_1 in $\mathbf{R}$ for every $r > 0$. On the other hand, we deduce from the condition imposed on g that for some r small enough, p_n maps M_2^r onto γ_2 which is either a compact interval of $\mathbf{R}$, or an arc in $\mathbf{C}$ that intersects γ_1 only at the origin. In both cases p maps $M_1^r \cup M_2^r$ onto a compact set γ in $\mathbf{C}$ which is simply connected and has empty interior.

To show that $\pi(p_n^{-1}(t) \cap (M_1^r \cup M_2^r))$ is simply connected, consider the equation $z^n + \bar{z}^n = t$. In polar coordinates, we have $\rho^n \cos(n\theta) = t/2$. It is now easy to see that $\pi(p_n^{-1}(t) \cap M_1^r)$ is simply connected and has empty interior. Next, from (3) we deduce

$$\pi(p_n^{-1}(t) \cap M_2^r) = \pi(p_n^{-1}(t') \cap M_1^r),$$

where $t' + g(t') = t$. Thus $\pi(p_n^{-1}(t) \cap M_2^r)$ is simply connected and $\pi(p_n^{-1}(t) \cap (M_1^r \cup M_2^r))$, being again the union of a disjoint family of simple arcs, is simply connected, without interior. $\square$

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REFERENCES

[D1] Paepe, P.J. de – Approximation on a disk, I. Math. Zeitschrift **212**, 145–152 (1993).

[D2] Paepe, P.J. de – Approximation on a disk, III. Indag. Math., N.S. **4**, 483–487 (1993).

[D3] Paepe, P.J. de – Approximation on a disk, IV. Indag. Math., N.S. **6**, 477–479 (1995).

[D4] Paepe, P.J. de – Algebras of continuous functions on disks. Proc. Royal Irish Acad. **96A**, 85–90 (1996).

[DO] Paepe, P.J. de and A.G. O'Farrell – Approximation on a disk, II Math. Zeitschrift **212**, 153–156 (1993).

[G] Gamelin, T. – Uniform Algebras. Prentice-Hall (1969).

[K] Kallin, E. – Fat polynomially convex sets. Function algebras (Proc. Intern. Symp Function Algebras, Tulane Univ., 1965). Chigago, Scott, Foresman, 149–152 (1966).

[MW] Moser, J. and S. Webster – Normal forms of real surfaces in $\mathbf{C}^2$ near complex tangents and hyperbolic surface transformation. Acta Math. **150**, 255–296 (1983).

[Sg1] Stolzenberg, G. – Polynomially and rationally convex sets. Acta Math. **109**, 259–289 (1963).

[Sg2] Stolzenberg, G. – Uniform approximation on smooth curves. Acta Math. **115**, 185–196 (1966).

[St] Stout, E. – The Theory of Uniform Algebras. Bogden and Quigley (1971).

[W] Weinstock, B – On the polynomial convexity of the union of two maximal totally real subspaces. Math. Ann. **282**, 131–138 (1988).

[We] Wermer, J. – Banach Algebras and Several Complex Variables. Second Edition, Springer-Verlag (1976).